# Mathematics: analysis and approaches
# Higher Level
# Paper 2

**WORKED SOLUTIONS**

2 hours

**Instructions to candidates**

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 7]

The circle shown below has center $\mathrm{O}$ and radius measuring 4.25 cm.

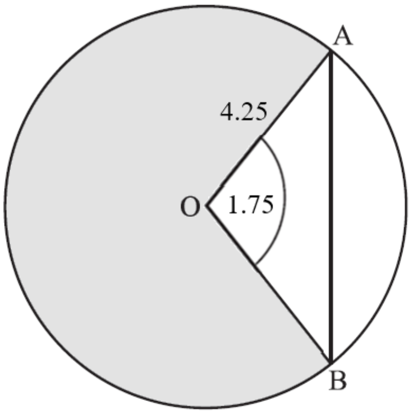



*diagram not to scale*

Points $\mathrm{A}$ and $\mathrm{B}$ lie on the circle and angle $\mathrm{AOB}$ measures 1.75 radians.

(a) Find $\mathrm{AB}$. [3]

(b) Find the area of the shaded region. [4]

(a) using the cosine rule:

$$\mathrm{AB}^2 = 4.25^2 + 4.25^2 - 2\cdot 4.25\cdot 4.25\cdot\cos(1.75)$$

$$\approx 42.5641\ldots$$

$$\mathrm{AB} = \sqrt{42.5641\ldots} \approx 6.5241\ldots$$

Thus, $\mathrm{AB} \approx 6.52\text{ cm}$

(b) angle corresponding to shaded sector $= 2\pi - 1.75$

area of shaded sector $= \frac{1}{2}r^2\theta$

$$= \frac{1}{2}\cdot 4.25^2\cdot(2\pi - 1.75)$$

$$\approx 40.9403\ldots$$

Thus, area of shaded sector $\approx 40.9\text{ cm}^2$

**2.** [Maximum mark: 6]

A multiple-choice test consists of 12 questions. Each question has four answers from which to choose. Only one of the answers is correct. For each question, Boris randomly chooses one of the four answers.

(a) Write down the expected number of questions Boris answers correctly. [1]

(b) Find the probability that Boris answers exactly three questions correctly. [2]

(c) Find the probability that Boris answers more than three questions correctly. [3]

(a) Using formula for the mean of a binomial distribution:

$\mathrm{E}(X) = np = 12 \cdot 0.25 = 3$

(b) Let $X$ be the random variable representing the number of questions Boris answers correctly

$X \sim \mathrm{B}(12, 0.25)$

Find $\mathrm{P}(X = 3)$ using GDC with $n = 12$, $p = 0.25$, lower bound $= 3$ and upper bound $= 3$

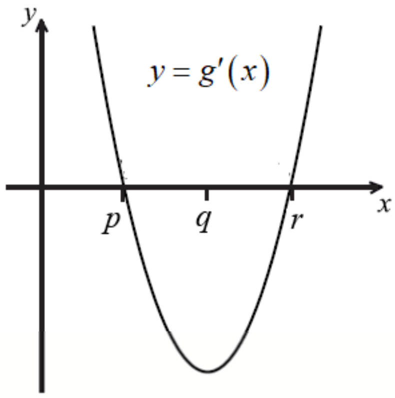
Thus, $\mathrm{P}(X = 3) \approx 0.258$

(c) Find $\mathrm{P}(X > 3)$ using GDC with $n = 12$, $p = 0.25$, lower bound $= 4$ and upper bound $= 12$

Thus, $\mathrm{P}(X > 3) \approx 0.351$

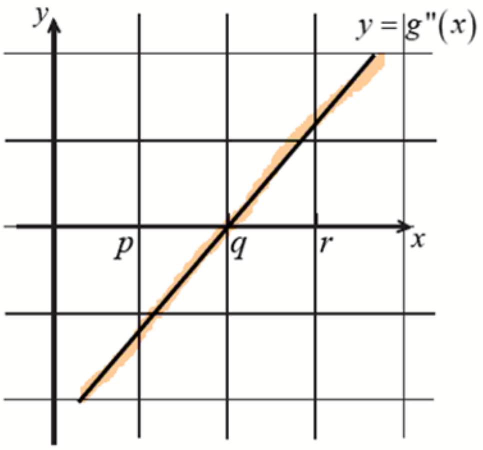
**3.** [Maximum mark: 6]

The diagram below shows part of the graph of the **gradient** function, $y = g'(x)$.




**Worked Solution**

(a) On the grid below, sketch a graph of $y = g''(x)$, clearly indicating the $x$-intercept. [2]




The graph of $y = g'(x)$ is a parabola, so $g'(x) = ax^2 + bx + c,\ a,b,c \in \mathbb{R}$. Therefore, $g''(x) = 2ax + b$. Hence, the graph of $y = g''(x)$ will be a straight line.

The graph of $y = g''(x)$ corresponds to the gradient of the graph of $y = g'(x)$, which is negative on the interval $-\infty < x < q$, zero at $x = q$, and positive on the interval $q < x < \infty$.

(b) Complete the table below, for the graph of $y = g(x)$. [2]

| | $x$-coordinate |
|---|---|
| (i) maximum point on $g$ | $x = p$ |
| (ii) minimum point on $g$ | $x = r$ |

(c) Justify your answer to part (b) (ii). [2]

Minimum point on the graph of $y = g(x)$ exists where $g'(x) = 0$ and $g''(x) > 0$.

These conditions are satisfied at $x = r$.

**4.** [Maximum mark: 7]

Given that events $A$ and $B$ are independent, $\mathrm{P}(B) = 2\mathrm{P}(A)$, and $\mathrm{P}(A \cup B) = 0.72$, find $\mathrm{P}(B)$.

For independent events, $\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$

Therefore, the formula $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ becomes

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A) \cdot \mathrm{P}(B)$

Substitute $\mathrm{P}(A \cup B) = 0.72$ and $\mathrm{P}(A) = \frac{1}{2}\mathrm{P}(B)$ into the equation above

$0.72 = \frac{1}{2}\mathrm{P}(B) + \mathrm{P}(B) - \frac{1}{2}\mathrm{P}(B)\mathrm{P}(B)$

$\frac{1}{2}[\mathrm{P}(B)]^2 - \frac{3}{2}\mathrm{P}(B) + 0.72 = 0$

Let $\mathrm{P}(B) = x$:

$\frac{1}{2}x^2 - \frac{3}{2}x + 0.72 = 0$

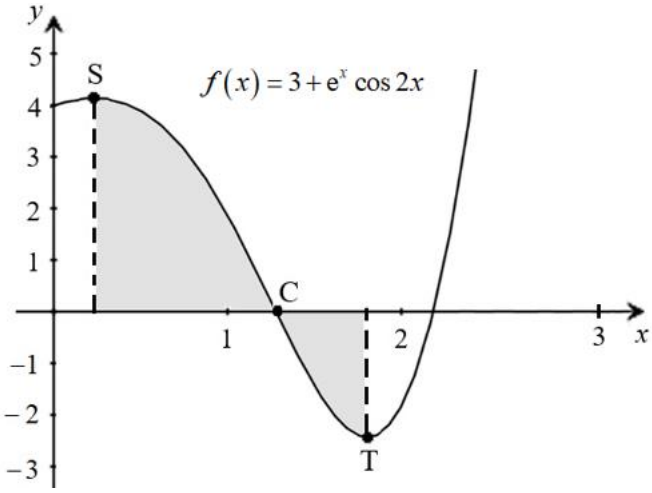
solve for $x$ on GDC

$x = 0.6$; also $x = 2.4$ but $0 \le \mathrm{P}(B) \le 1$, hence $x \ne 2.4$

Thus, $\mathrm{P}(B) = 0.6$

**5.** [Maximum mark: 6]

Let $f(x) = 3 + e^x \cos 2x$, for $0 \le x \le 3$. A portion of the graph of $f$ is shown below.




There is an $x$-intercept at the point C, a local maximum point at S where $x = s$, and a minimum point at T where $x = t$.

(a) Write down the following:

(i) the $x$-coordinate of C;

(ii) the value of $s$;

(iii) the value of $t$. [3]

(b) (i) Let $\int_s^t f(x)\,dx = k$. Calculate the value of $k$.

(ii) Explain why $k$ is **not** the area of the shaded region. [3]

(a) (i) $x$-coordinate of C exists where $f(x) = 0$

$3 + e^x \cos 2x = 0$; solve for $x$ on GDC

$x \approx 1.2792\ldots$

Thus, the $x$-coordinate of C is $x \approx 1.28$ (3 sf)

(ii) Find value of $s$ by analysing maximum points on the graph of $f(x)$ using GDC

$s \approx 0.232$ (3 sf)

(iii) Find value of $t$ by analysing minimum points on the graph of $f(x)$ using GDC

$t \approx 1.80$ (3 sf)

**Question 5 continues on the next page**

**Question 5 continued**

(b) (i) Write down integral:

$$\int_{0.2318\ldots}^{1.802\ldots} \left(3+e^{x}\cos(2x)\right)dx = k$$

Evaluate the integral on GDC

$$\int_{0.2318\ldots}^{1.802\ldots} \left(3+e^{x}\cos(2x)\right)dx \approx 2.0912.$$

Thus, $k \approx 2.09$ (3 sf)

(ii) The region between $f$ and the $x$-axis from $x = s$ to $x = c$ is **above** the $x$-axis and the region between $f$ and the $x$-axis from $x = c$ to $x = t$ is **below** the $x$-axis. Hence, the definite integral from $x = s$ to $x = c$ is a **positive** number and the definite integral from $x = c$ to $x = t$ is a **negative** number. Therefore, since $k$ is the sum of the two definite integrals then the value of $k$ is less than the total area of the shaded region.

**6.** [Maximum mark: 5]

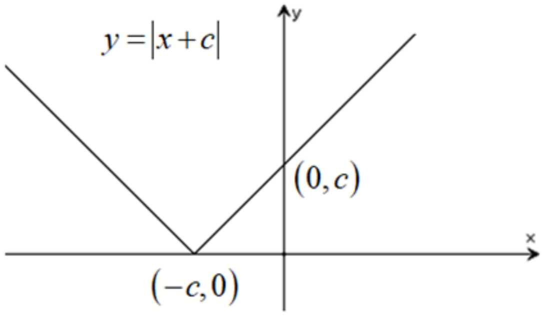
Given that $c > 0$, find the value(s) of $x$ that solve the equation $|x+c| = |x| + c$.

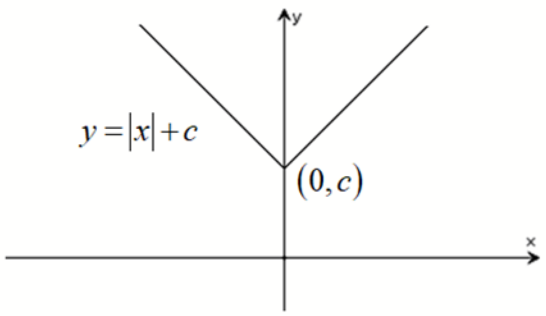
Let $f(x)$ be the function defined by $f(x) = |x|$

The graph of the function $y = |x+c|$ is achieved by translating the graph of $f(x)$ by $c$ units to the left given that $c > 0$.

The graph of the function $y = |x| + c$ is achieved by translating the graph of $f(x)$ upwards by $c$ units given that $c > 0$.

The graphs of $y = |x+c|$ and $y = |x| + c$ are shown below.



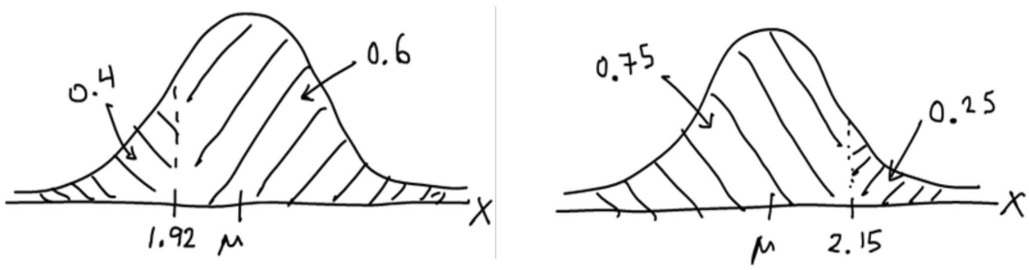




As $y = |x+c|$ and $y = |x| + c$ are simply horizontal and vertical shifts of the same absolute value function $f(x) = |x|$, it is clear that the graphs of the two functions intersect for all $x \geq 0$

Thus, $|x+c| = |x| + c$ for $x \geq 0, x \in \mathbb{R}$

**7.** [Maximum mark: 6]

The heights of sunflower plants in a large field can be modelled by a normal distribution. It is given that 60% of the plants are taller than 1.92 m and 25% are taller than 2.15 m. Find the mean and the standard deviation of the heights of the plants.




Let $X$ be the random variable representing the height of a sunflower plant

$$P(X > 1.92) = 0.6 \Rightarrow P(X < 1.92) = 0.4 \Rightarrow Z = -0.2533\ldots$$

$$P(X > 2.15) = 0.25 \Rightarrow P(X < 2.15) = 0.75 \Rightarrow Z = 0.6745\ldots$$

Using the formula for the standardised normal variable:

$$z = \frac{x-\mu}{\sigma} \Rightarrow \begin{cases} -0.2533\ldots = \dfrac{1.92-\mu}{\sigma} \\ 0.6745\ldots = \dfrac{2.15-\mu}{\sigma} \end{cases}$$

$$\begin{cases} -0.2533\sigma = 1.92-\mu \\ 0.6744\sigma = 2.15-\mu \end{cases}$$

Solving system of equations on GDC

$\mu = 1.9828\ldots,\ \sigma = 0.2479\ldots$

Thus, $\mu \approx 1.98\,\text{m}$ and $\sigma \approx 0.248\,\text{m}$

**8.** [Maximum mark: 7]

Find the three cube roots of $4\sqrt{3}-4\mathrm{i}$ and express them exactly in exponential form, $r\,\mathrm{e}^{\mathrm{i}\theta}$.

$z=4\sqrt{3}-4\mathrm{i}$ ; express $z$ in the form $r\operatorname{cis}\theta$ where $r=|z|$ and $\theta=\arg(z)$

$$r=|z|=\sqrt{\left(4\sqrt{3}\right)^2+(-4)^2}=\sqrt{64}=8$$

$$\theta=\arg(z)=\arctan\left(\frac{-4}{4\sqrt{3}}\right)=\arctan\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$$

So, $z=8\operatorname{cis}\left(-\frac{\pi}{6}\right)$

Let $z_1$, $z_2$ and $z_3$ be the cube roots of $z$

$$(r\operatorname{cis}\theta)^3=8\operatorname{cis}\left(-\frac{\pi}{6}+k\cdot 2\pi\right),\ k\in\mathbb{Z}$$

By de Moivre's theorem:

$$r\operatorname{cis}\theta=8^{\frac{1}{3}}\operatorname{cis}\left(-\frac{\pi}{18}+k\cdot\frac{2\pi}{3}\right)=2\operatorname{cis}\left(-\frac{\pi}{18}+k\cdot\frac{12\pi}{18}\right)$$

For $k=0$: $z_1=2\operatorname{cis}\left(-\frac{\pi}{18}\right)=2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{18}\right)}$

For $k=1$: $z_2=2\operatorname{cis}\left(\frac{11\pi}{18}\right)=2\mathrm{e}^{\mathrm{i}\left(\frac{11\pi}{18}\right)}$

For $k=2$: $z_3=2\operatorname{cis}\left(\frac{23\pi}{18}\right)=2\operatorname{cis}\left(-\frac{13\pi}{18}\right)=2\mathrm{e}^{\mathrm{i}\left(-\frac{13\pi}{18}\right)}$

**9.** [Maximum mark: 6]

The equations of three planes are given by
$$\begin{aligned} x+2y+z&=3 \\ -x+2y+3z&=1 \\ -2x+y+3z&=a \end{aligned}$$

(a) Find the value of $a$ such that the three planes intersect in one line. [4]

(b) Find a vector equation for the line of intersection. [2]

(a)
$$\begin{aligned} x+2y+z&=3 \\ -x+2y+3z&=1 \\ -2x+y+3z&=a \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -1 & 2 & 3 & 1 \\ -2 & 1 & 3 & a \end{array}\right) \begin{matrix}(1)\\(2)\\(3)\end{matrix}$$

$(1)+(2)\to(2),\ 2(1)+(3)\to(3)$ $\qquad$ $(2)\div 4\to(2)$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 5 & 5 & 6+a \end{array}\right) \begin{matrix}(1)\\(2)\\(3)\end{matrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 5 & 5 & 6+a \end{array}\right) \begin{matrix}(1)\\(2)\\(3)\end{matrix}$$

$(3)-5(2)\to(3)$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & a+1 \end{array}\right) \begin{matrix}(1)\\(2)\\(3)\end{matrix}$$

Thus, the three planes intersect in one line (infinite solutions) when $a+1=0 \Rightarrow a=-1$

(b) from part (a):

$$\begin{aligned} x+2y+z&=3 \quad (1) \\ y+z&=1 \quad (2) \end{aligned}$$

from (2), $y=1-z$

let $z=\lambda$, then $y=1-\lambda$

substituting into (1) gives:

$$x+2(1-\lambda)+\lambda=3 \Rightarrow x=1+\lambda$$

Hence, parametric equations for the line of intersection are $\begin{cases} x=1+\lambda \\ y=1-\lambda \\ z=\lambda \end{cases}$

Thus, equation for line of intersection in vector form is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

## Section B

Answer **all** the questions on the answer sheets provided. Please start each question on a new page.

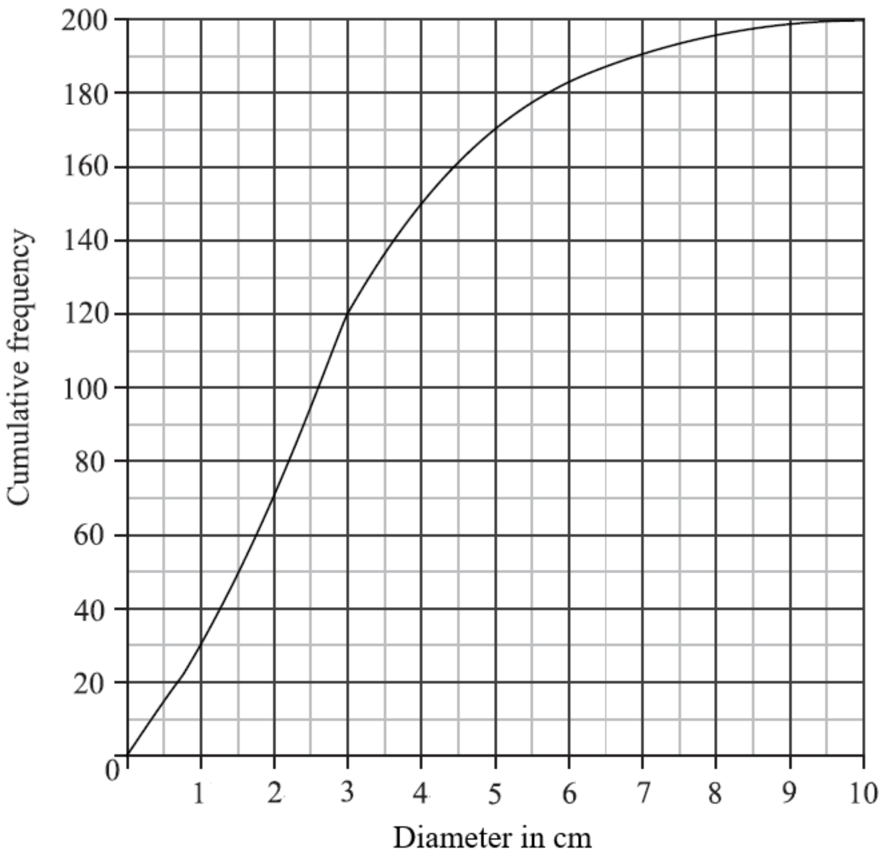
**10.** [Maximum mark: 15]

A farmer has an operation growing button mushrooms indoors that are sold at a local market. On a particular day, the farmer harvests 200 button mushrooms and measures the diameter ($d$) of each mushroom in centimeters. The results are shown in the frequency table below.

| diameter, $d$ cm | $0 < d \le 1$ | $1 < d \le 2$ | $2 < d \le 3$ | $3 < d \le 4$ | $4 < d \le 6$ | $6 < d \le 7.5$ | $7.5 < d \le 10$ |
|---|---|---|---|---|---|---|---|
| frequency | 30 | 40 | 50 | 30 | 33 | 11 | 6 |

(a) Calculate an estimate for the mean of the diameters of the mushrooms. [3]

(b) A cumulative frequency graph is given below for the diameters of the mushrooms.




Use the graph to answer the following.

(i) Estimate the interquartile range.

(ii) Given that 20% of the mushrooms have a diameter more than $k$ cm, find the value of $k$. [6]

**Question 10 continues on the next page**

**Question 10 continued**

In preparation for selling the mushrooms, the farmer classifies each of them as *small*, *medium* or *large* using the following criteria.

*Small*: diameter is less than 2 cm
*Medium*: diameter is greater than or equal to 2 cm but less than 6 cm
*Large*: diameter is greater than or equal to 6 cm

(c) Write down the probability that a mushroom randomly selected from the day's harvest is *Small*. [2]

The cost of a *Small* mushroom is $\$0.10$, a *Medium* mushroom is $\$0.15$ and a *Large* mushroom is $\$0.25$.

(d) Copy and complete the table below which is the probability distribution for the cost $\$X$. [2]

| Cost $\$X$ | 0.10 | 0.15 | 0.25 |
|---|---|---|---|
| $\mathrm{P}(X=x)$ | | 0.565 | |

(e) Find $\mathrm{E}(X)$. [2]

**Worked Solution:**

(a) mean $=\dfrac{\sum_{i=1}^{7}(\bar{d}_i\cdot f_i)}{n}$, where $\bar{d}_i$ is the midpoint of the $i$th diameter interval, $f_i$ is the interval's corresponding frequency, and $n$ is the total number of mushrooms

$$\text{mean} = \frac{0.5\cdot30+1.5\cdot40+2.5\cdot50+3.5\cdot30+5\cdot33+6.75\cdot11+8.75\cdot6}{200}\approx 2.9837\ldots\approx 2.98\text{ cm}$$

(b) (i) $\mathrm{IQR}=\mathrm{Q3}-\mathrm{Q1}$

Q1 is diameter corresponding to a cumulative frequency of $\frac{200}{4}=50$; from the graph, $\mathrm{Q1}=1.5$

Q3 is diameter corresponding to a cumulative frequency of $3\cdot\frac{200}{4}=150$; from the graph, $\mathrm{Q3}=4$

Therefore, $\mathrm{IQR}=4-1.5=2.5$

(ii) 20% of 200 is 40; $200-40=160$; so looking for the diameter which 160 of the mushrooms have less than. The diameter corresponding to a cumulative frequency of 160 is $4.5\,\text{cm}$. Thus, $k=4.5$

solution continued on next page >>

**10.** solution continued

(c) $\mathrm{P}(\text{small}) = \mathrm{P}(\text{diameter} < 2)$; a diameter of $2\,\text{cm}$ corresponds to a cumulative frequency of 70

$\frac{70}{200} = 0.35$; Thus, $\mathrm{P}(\text{small}) = 0.35$

(d)

| Cost $\$X$ | 0.10 | 0.15 | 0.25 |
|---|---|---|---|
| $\mathrm{P}(X = x)$ | 0.35 | 0.565 | 0.085 |

$\mathrm{P}(X = 0.10) = \mathrm{P}(\text{small}) = 0.35$

$\mathrm{P}(X = 0.25) = 1 - (\mathrm{P}(X = 0.10) + \mathrm{P}(X = 0.15)) = 1 - (0.35 + 0.565) = 0.085$

(e) $\mathrm{E}(X) = \sum x\mathrm{P}(X = x) = 0.10 \cdot 0.35 + 0.15 \cdot 0.565 + 0.25 \cdot 0.085$

Thus, $\mathrm{E}(X) \approx \$0.141$

**11.** [Maximum mark: 23]

The Cartesian equation of line $L_1$ is $x-5=\dfrac{y+3}{-3}=\dfrac{z-4}{2}$ and the Cartesian equation of line $L_2$ is $\dfrac{x-2}{2}=y+1=\dfrac{z-3}{-1}$.

(a) Lines $L_1$ and $L_2$ intersect at point P. Find the coordinates of P. [5]

(b) Point $Q$ is the point on $L_1$ that is nearest to the origin. Find the **exact** coordinates of $Q$. [6]

(c) Determine a Cartesian equation for the plane that contains $L_1$ and $L_2$. [4]

(d) Find the degree measure of the acute angle between the lines $L_1$ and $L_2$. [4]

(e) Line $L_3$ has the vector equation $\vec{\mathbf{r}}=\begin{pmatrix}2\\-3\\1\end{pmatrix}+t\begin{pmatrix}k\\-2\\4\end{pmatrix}$. Find the values of $k$ such that the angle between $L_2$ and $L_3$ is $60°$. [4]

**Worked Solution:**

(a) Express $L_1$ and $L_2$ in vector form

$$L_1:\ x-5=\frac{y+3}{-3}=\frac{z-4}{2}=\lambda \;\Rightarrow\; \begin{matrix}x=5+\lambda\\ y=-3-3\lambda\\ z=4+2\lambda\end{matrix} \;\Rightarrow\; \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\-3\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\2\end{pmatrix}$$

$$L_2:\ \frac{x-2}{2}=y+1=\frac{z-3}{-1}=\mu \;\Rightarrow\; \begin{matrix}x=2+2\mu\\ y=-1+\mu\\ z=3-\mu\end{matrix} \;\Rightarrow\; \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\-1\\3\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$\begin{aligned} x=5+\lambda=2+2\mu &\;\Rightarrow\; \lambda-2\mu=-3 \quad (1)\\ y=-3-3\lambda=-1+\mu &\;\Rightarrow\; -3\lambda-\mu=2 \quad (2)\\ z=4+2\lambda=3-\mu &\;\Rightarrow\; 2\lambda+\mu=-1 \quad (3)\end{aligned}$$

(2) + (3) gives $-\lambda=1 \Rightarrow \lambda=-1$

Substituting into (1) gives $-1-2\mu=-3 \Rightarrow \mu=1$

Check that these values of $\lambda$ and $\mu$ generate same point for both $L_1$ and $L_2$.

$$L_1:\ \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\-3\\4\end{pmatrix}+(-1)\begin{pmatrix}1\\-3\\2\end{pmatrix}=\begin{pmatrix}4\\0\\2\end{pmatrix} \qquad L_2:\ \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\-1\\3\end{pmatrix}+(1)\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}4\\0\\2\end{pmatrix}$$

Thus, $L_1$ and $L_2$ intersect at $\text{P}(4,\ 0,\ 2)$

(b) As Q is the point on $L_1$ nearest the origin, the vector $\overrightarrow{OQ}$ will be perpendicular to $L_1$.

Any point on $L_1$ can be expressed as $(5+\lambda,\ -3-3\lambda,\ 4+2\lambda)$; hence, $\overrightarrow{OQ}=\begin{pmatrix}5+\lambda\\-3-3\lambda\\4+2\lambda\end{pmatrix}$.

A direction vector for $L_1$ is $\begin{pmatrix}1\\-3\\2\end{pmatrix}$. Find $\lambda$ such that dot product of $\overrightarrow{OQ}$ and $\begin{pmatrix}1\\-3\\2\end{pmatrix}$ is 0.

$$\begin{pmatrix}5+\lambda\\-3-3\lambda\\4+2\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\2\end{pmatrix}=5+\lambda 9+9\lambda+8+4\lambda=0 \ \Rightarrow\ 22+14\lambda=0\ \Rightarrow\ \lambda=-\frac{11}{7}$$

Substitute into general expression for any point on $L_1$ to obtain coordinates of Q.

$$\left(5+\left(-\frac{11}{7}\right),\ -3-3\left(-\frac{11}{7}\right),\ 4+2\left(-\frac{11}{7}\right)\right);\ \text{thus, coordinates of Q are } \left(\frac{24}{7},\ \frac{12}{7},\ \frac{6}{7}\right)$$

(c) vector product of direction vectors for $L_1$ & $L_2$: $\begin{pmatrix}1\\-3\\2\end{pmatrix}\times\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}(-3)\cdot(-1)-2\cdot 1\\2\cdot 2-1\cdot(-1)\\1\cdot 1-(-3)\cdot 2\end{pmatrix}=\begin{pmatrix}1\\5\\7\end{pmatrix}$

Hence, Cartesian equation for plane containing $L_1$ and $L_2$ is $x+5y+7z=d$.

Using the position vector of $L_2$ as a point in the plane: $d=2+5(-1)+7(3)=18$

Thus, a Cartesian equation for the plane containing $L_1$ and $L_2$ is $x+5y+7z=18$

(d) The angle, $\theta$, between vectors $\mathbf{v}$ and $\mathbf{w}$ is given by $\cos\theta=\dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\ \Rightarrow\ \theta=\cos^{-1}\left(\dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\right)$

Hence, angle between $L_1$ and $L_2$ is $\theta=\cos^{-1}\left(\dfrac{\begin{pmatrix}1\\-3\\2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}}{\left|\begin{pmatrix}1\\-3\\2\end{pmatrix}\right|\left|\begin{pmatrix}2\\1\\-1\end{pmatrix}\right|}\right)=\cos^{-1}\left(\dfrac{-3}{\sqrt{14}\sqrt{6}}\right)\approx 109.11...^\circ$

Thus, the acute angle between $L_1$ and $L_2$ is $180^\circ-109.11^\circ\approx 70.9^\circ$

(e) acute angle between $L_2$ and $L_3$ is also $60^\circ$ when $\theta=120^\circ$; hence, $\cos\theta=\pm\dfrac{1}{2}$

$$\cos\theta=\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\ \Rightarrow\ \cos(60^\circ)=\frac{\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}k\\-2\\4\end{pmatrix}}{\left|\begin{pmatrix}2\\1\\-1\end{pmatrix}\right|\left|\begin{pmatrix}k\\-2\\4\end{pmatrix}\right|}=\frac{2k-6}{\sqrt{6}\sqrt{k^2+20}}=\pm\frac{1}{2}$$

Solving for $k$ on GDC gives $k\approx 9.3431\ldots$ or $k\approx 0.25687\ldots$; Thus, $k\approx 9.34$ or $k\approx 0.257$

**12.** [Maximum mark: 16]

(a) Show that $\frac{1}{4-x^2}$ can be expressed as $\frac{1}{4(x+2)}-\frac{1}{4(x-2)}$. [4]

(b) Hence, find $\int \frac{1}{4-x^2}\,dx$. [4]

The region $R$ is bounded by the graph of $h(x)=\frac{1}{4-x^2}$ and the line $y=\frac{4}{7}$.

(c) Find the **exact** area of $R$. [5]

(d) The line $y=m$, where $m\in\mathbb{R}$, divides $R$ into two regions of equal area. Write an equation whose solution is the value of $m$. Do **not** solve the equation. [3]

**Worked Solution:**

(a) $\frac{1}{4-x^2}=\frac{-1}{x^2-4}=\frac{-1}{(x+2)(x-2)}$

Express as the sum of two fractions

$\frac{-1}{(x+2)(x-2)}=\frac{A}{x+2}+\frac{B}{x-2},\ A,B\in\mathbb{R}$

$\frac{-1}{(x+2)(x-2)}=\frac{A(x-2)}{(x+2)(x-2)}+\frac{B(x+2)}{(x+2)(x-2)}$

Equate numerators: $-1=A(x-2)+B(x+2)$

Let $x=2$: $-1=A(2-2)+B(2+2)\ \Rightarrow\ -1=4B\ \Rightarrow\ B=-\frac{1}{4}$

Let $x=-2$: $-1=A(-2-2)+B(-2+2)\ \Rightarrow\ -1=-4A\ \Rightarrow\ A=\frac{1}{4}$

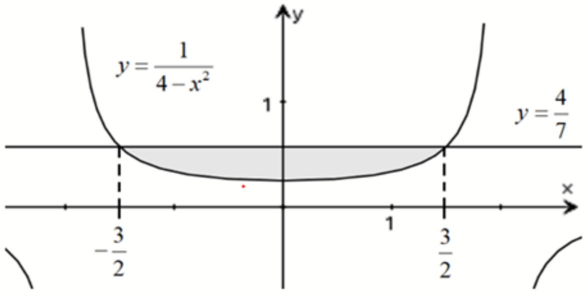
Thus, $\frac{1}{4-x^2}=\frac{1}{4(x+2)}-\frac{1}{4(x-2)}$ **Q.E.D.**

(b) $\int\frac{1}{4-x^2}\,dx=\int\left(\frac{1}{4(x+2)}-\frac{1}{4(x-2)}\right)dx=\frac{1}{4}\int\frac{1}{x+2}\,dx-\frac{1}{4}\int\frac{1}{x-2}\,dx$

Thus, $\int\frac{1}{4-x^2}\,dx=\frac{1}{4}\ln|x+2|-\frac{1}{4}\ln|x-2|+C$

(c) Find intersection points:

$$\frac{1}{4-x^2}=\frac{4}{7} \Rightarrow 4-x^2=\frac{7}{4} \Rightarrow x^2=\frac{9}{4} \Rightarrow x=\pm\frac{3}{2}$$




Because of the symmetry of region $R$ (shaded in diagram above):

$$\text{area of } R = 2\int_0^{\frac{3}{2}}\left(\frac{4}{7}-\frac{1}{4-x^2}\right)dx=\left[\frac{4}{7}x-\frac{1}{4}\ln|x+2|+\frac{1}{4}\ln|x-2|\right]_0^{\frac{3}{2}}$$

$$=2\left\{\left(\frac{4}{7}\left(\frac{3}{2}\right)-\frac{1}{4}\ln\left|\frac{3}{2}+2\right|+\frac{1}{4}\ln\left|\frac{3}{2}-2\right|\right)-\left(\frac{4}{7}(0)-\frac{1}{4}\ln|0+2|+\frac{1}{4}\ln|0-2|\right)\right\}$$

$$=2\left\{\left(\frac{6}{7}-\frac{1}{4}\ln\left(\frac{7}{2}\right)+\frac{1}{4}\ln\left(\frac{1}{2}\right)\right)-\left(0-\frac{1}{4}\ln(2)+\frac{1}{4}\ln(2)\right)\right\}$$

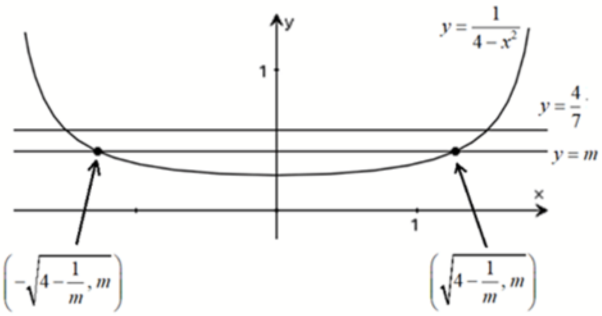
$$=\frac{12}{7}-\frac{1}{2}\ln\left(\frac{7}{2}\right)+\frac{1}{2}\ln\left(\frac{1}{2}\right)=\frac{12}{7}+\frac{1}{2}\left[\ln\left(\frac{1}{2}\right)-\ln\left(\frac{7}{2}\right)\right]$$

$$=\frac{12}{7}+\frac{1}{2}\ln\left(\frac{1}{7}\right)=\frac{24+7\ln\left(\frac{1}{7}\right)}{14}=\frac{24+7\ln(1)-7\ln(7)}{14}=\frac{24-7\ln(7)}{14}$$

Thus, area of $R=\dfrac{24-7\ln 7}{14}$ units$^2$ $\left[\text{or } \dfrac{12}{7}+\dfrac{1}{2}\ln\left(\dfrac{1}{7}\right) \text{ units}^2\right]$

(d) Find $x$-coordinates of intersection points of $y=\dfrac{1}{4-x^2}$ and horizontal line $y=m$.

$$\frac{1}{4-x^2}=m \Rightarrow \frac{1}{m}=4-x^2 \Rightarrow x^2=4-\frac{1}{m} \Rightarrow x=\pm\sqrt{4-\frac{1}{m}}$$




Thus, $$\int_{-\sqrt{4-\frac{1}{m}}}^{\sqrt{4-\frac{1}{m}}}\left(m-\frac{1}{4-x^2}\right)dx=\frac{1}{2}\left(\frac{24-7\ln 7}{14}\right)=\frac{24-7\ln 7}{28}$$

or, from symmetry of region, $$\int_0^{\sqrt{4-\frac{1}{m}}}\left(m-\frac{1}{4-x^2}\right)dx=\frac{1}{4}\left(\frac{24-7\ln 7}{14}\right)=\frac{24-7\ln 7}{56}$$